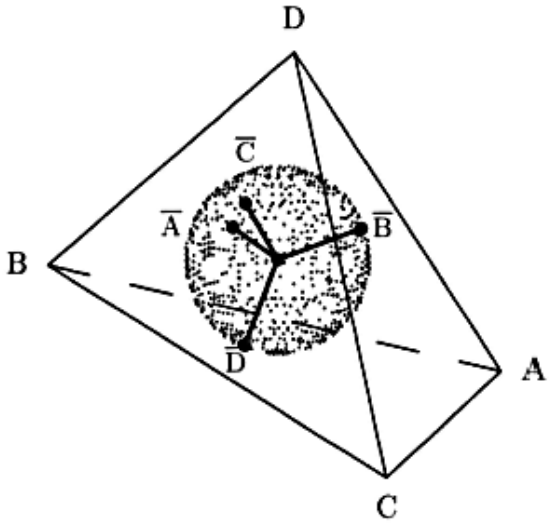
CHAPTER 20

On to Solids

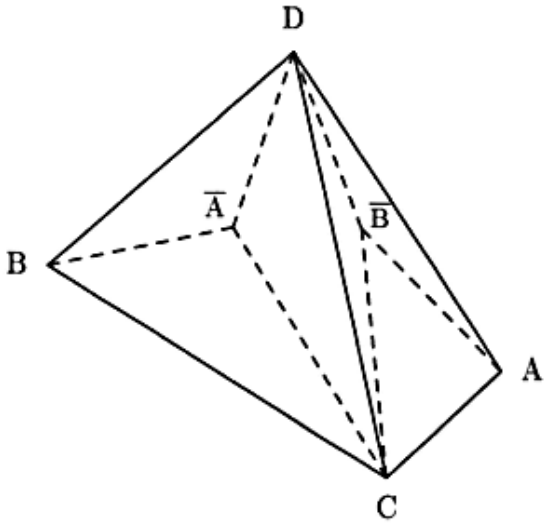
"Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries?" So, stirringly, began David Hilbert at the beginning of the twentieth century. His wide-ranging list of twenty-three problems, heralded before the International Congress of Mathematicians in Paris, has inspired a stampede of research during this, the first of his future centuries. And the exciting news was that the third problem on his list concerned dissections!

It sounded wonderful, but there was a flip side: First, Hilbert proposed a negative result, challenging mathematicians to prove the impossibility of certain three-dimensional dissections. Second, the problem was solved before the congress even got under way. Third, the problem has since been viewed as one motivated primarily by pedagogical concerns. So much for geometric dissection's brief membership in the vanguard of twentieth-century mathematics!

What a letdown! And although three-dimensional dissections were only briefly in that brightest of spotlights, we shall find that there have been some nice examples produced even in the shadows. We start our story a half century before dissection's fifteen minutes of fame. In an exchange of letters between Carl Friedrich Gauss and Christian Ludwig Gerling in 1844, Gauss had lamented that the only known procedure for proving the equality of the volume of a polyhedron with the volume of its mirror image was a method of exhaustion. However, in his return letter, Gerling



20.1: Inscribed sphere



20.2: Cutting tetrahedron

described a simple proof that established that an irregular tetrahedron has the same volume as its mirror image. Since a polyhedron can be split into tetrahedra, the result followed. Gerling gave a simple dissection of the tetrahedron into twelve pieces that can be reassembled into the mirror image. His dissection uses a circumscribed sphere, but "negative pieces" arise when the center of the sphere falls outside the tetrahedron.

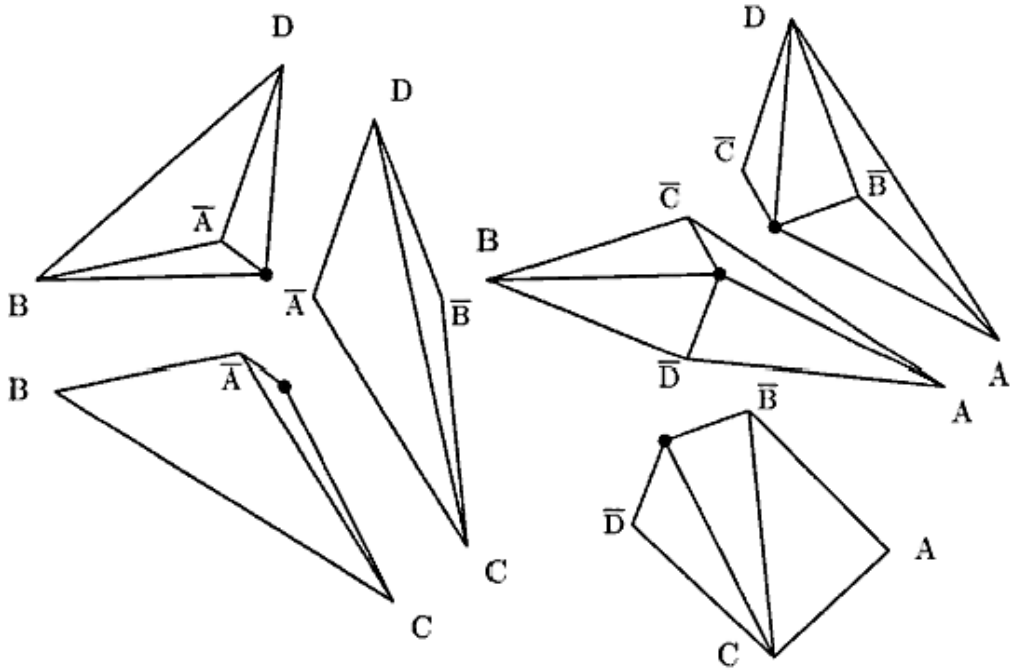
Juel (1903) gave a related 12-piece dissection that uses the inscribed sphere. Børge Jessen (1968) combined pairs of pieces in Juel's dissection to give a 6-piece dissection (Figure 20.3). It is easy to produce this three-dimensional analogue of the dissection in Figure 3.9: First, inscribe a sphere in the tetrahedron (Figure 20.1). Large dots represent the center of the sphere and the points of tangency. The point

Christian Ludwig Gerling was born in Hamburg in 1788 and died in Marburg in 1864. As a student of Carl Friedrich Gauss at Göttingen, he earned a doctorate in 1812 for studying the orbit of the asteroid Vesta. He taught at the lyceum in Kassel for five years, until he was appointed professor of mathematics, physics, and astronomy at Marburg, where he remained for the rest of his life.

Gerling was one of the few of Gauss's students who kept up a cordial and scientifically fruitful relationship with Gauss. He published many papers on questions in astronomy, such as the determination of the sun's parallax, the determination of the brightness of stars, and the measurement of the earth's magnetic variations. He also wrote a practical text on the application of least squares to geodesic questions.

of tangency with the face opposite vertex A is $\overline{A}$, with $\overline{B}$, $\overline{C}$, and $\overline{D}$ similarly named. Thick lines indicate the radii from the center to these points.

Next, cut the tetrahedron into six pieces. Base each piece on one edge of the tetrahedron, the points of tangency on the two faces that contain that edge in their boundaries, and the center of the inscribed sphere. The dashed lines in Figure 20.2 indicate the position of the cuts when the six pieces of the tetrahedron are assembled. In Figure 20.3 a large dot indicates the vertex in each piece that corresponds to the center, and each remaining vertex is labeled with the point in Figure 20.1 with which it corresponds. We can move the pieces in Figure 20.3 by translation



20.3: Irregular tetrahedron to its mirror image

back into the tetrahedron in Figure 20.2. Each piece has five vertices, some obscured from our view. Further, each piece has a plane of symmetry that contains its edge from the tetrahedron plus the center of the inscribed sphere. Since the mirror image tetrahedron will have a dissection to the same set of six pieces, we get the desired dissection. There is a simple generalization to any polyhedron in which a sphere can be inscribed, and the number of pieces equals the number of edges in the polyhedron.

Five years before Hilbert announced his list of problems, M. J. M. (Micaiah John Muller) Hill (1896) had taken a constructive approach. Hill, a professor of mathematics at the University College in London, had identified three infinite classes

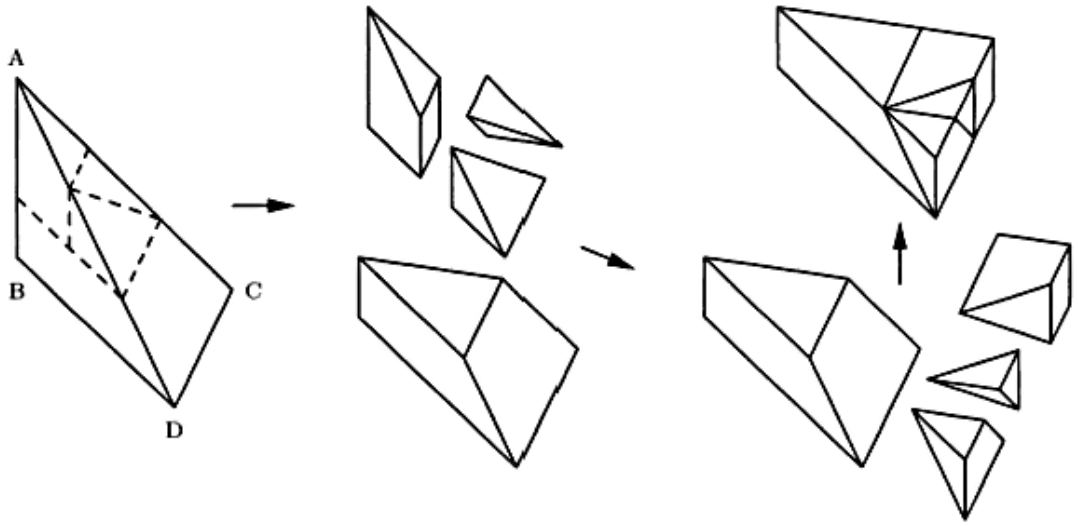
of tetrahedra whose volumes could be determined without resort to a method of limits. Any tetrahedron in his classes can be converted to a cube by dissection, and hence to any other tetrahedron in his classes.

We can derive Hill's first type of tetrahedron from a *rhombohedron*, which is a polyhedron with six faces, each of which is a rhombus. (We can produce the vertex-edge skeleton of a rhombohedron from the vertex-edge skeleton of a cube by moving two diagonally opposite vertices A and D either toward each other or away from each other. With the edges rigid and the vertices like universal joints, the skeleton will expand or contract like an accordion.) The rhombohedron decomposes into six tetrahedra, three identical and three mirror images. Each tetrahedron will contain the two vertices A and D, and two other vertices B and C such that AB, BC, and CD are edges of the rhombohedron. For a parameter r, with $0 < r < \sqrt{2}$, and a scaling coefficient a, the Hill tetrahedron of the first type will have edges AB, BC, and CD of length $a\sqrt{1 + r^2}$; edges AC and BD of length $2a$; and edge AD of length $a\sqrt{9 - 3r^2}$. Actually, this first type of tetrahedron, and the decomposition of a rhombohedron into six such tetrahedra, were described already by René Haüy (1801). Thus these could rightly be called Haüy-tetrahedra, which Hill then rediscovered.

Hill's second and third types of tetrahedron can easily be derived from the first. Let E be the midpoint of edge AD in a Hill tetrahedron of the first type. Then the tetrahedron with vertices A, B, C, and E is a Hill tetrahedron of the second type. Let F be the midpoint of edge BC in a Hill tetrahedron of the first type. Then the tetrahedron with vertices A, B, D, and F is a Hill tetrahedron of the third type.

At the time when Hill defined his tetrahedra, they seemed to be the exception, rather than the rule, with respect to dissection to a cube. For example, no one knew at that time how to dissect a regular tetrahedron to a cube in a finite number of pieces. (In fact, it can't be done.) Bricard (1896) had already attempted a proof of the conjecture that was embodied in Hilbert's third problem. Although incorrect, the attempted proof contained crucial elements of the eventual proof by Dehn (1900). The proof involves defining for each polyhedron what is now called its *Dehn invariant*, a sum of terms each of which is the length of an edge times a certain function of the angle of the two faces meeting at that edge. Dehn completed the proof by identifying a pair of polyhedra such that their Dehn invariants are different. Jean-Pierre Sydler (1965) showed that the condition of Dehn invariants was not only necessary but also sufficient. In other words, one polyhedron is dissectable to another if and only if for all appropriate functions, their Dehn invariants are equal. Disappointingly, no regular polyhedron is dissectable to any other regular polyhedron.

M. J. M. Hill did not supply dissections to illustrate the decomposability of any of his tetrahedra to a cube. The Swiss did this more than a half-century later. First,

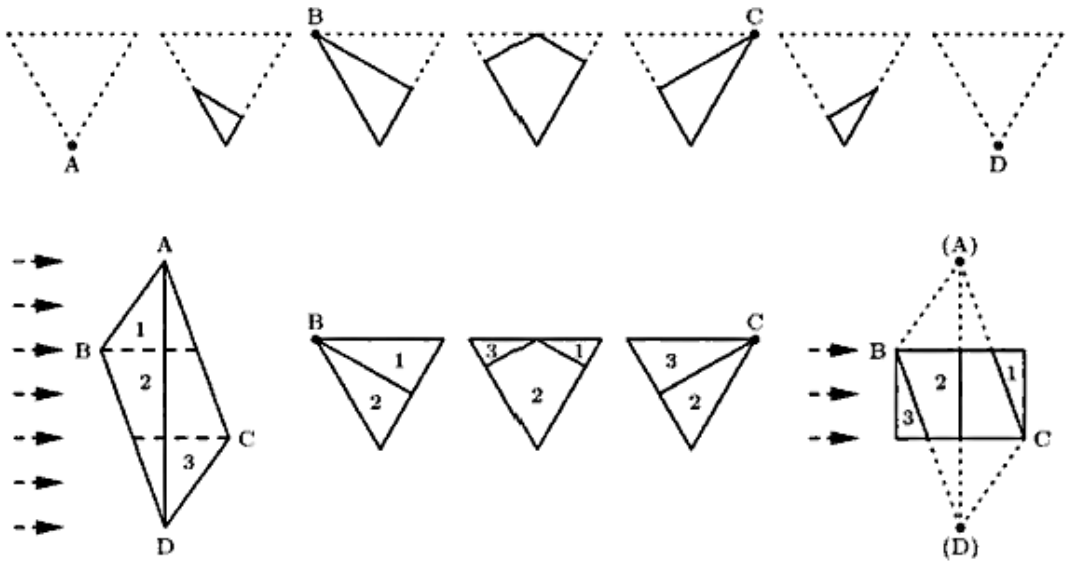


20.4: Sydler's Hill tetrahedron to isosceles triangular prism

Sydler (1956) demonstrated how to dissect a particular Hill tetrahedron of the first type to a right prism whose base is an isosceles right triangle. Once we have a prism, it is easy to convert the figure to a rectangular block, and hence to a cube. For this example, $r = 1$, so that the dihedral angles at edges AC, BC, and BD are 90°; that implies that edges AB and CD are perpendicular to edge BC. The corresponding rhombohedron is precisely the cube. Sydler cut this tetrahedron into four pieces that rearrange to form a prism, where the top face of the prism is an isosceles right triangle (Figure 20.4). An analogous dissection applies for any tetrahedron of Hill's first type, giving a prism that is not a right prism. It is easy to convert such a prism to a right prism.

Remarkably, there is an even nicer dissection of that tetrahedron to a different triangular prism. Philipp Schöbi (1985), and independently Anton Hanegraaf, found

Jean-Pierre Sydler was born in Neuenburg, Switzerland, in 1921. At the Eidgenössische Technische Hochschule (ETH), Zurich, he earned a graduate diploma in mathematics in 1943, and a Dr. sc. math. in 1946. In 1950 he left his position in the mathematics department to join the staff of the library at ETH. There he continued to publish mathematical papers in his spare time. In 1966 he was awarded the gold medal of the Royal Danish Academy of Sciences and Letters for resolving Hilbert's third problem. Becoming director of the library in 1963, Sydler pioneered the use of automation in the library system. He served as director until his retirement in 1986. He died in 1988.

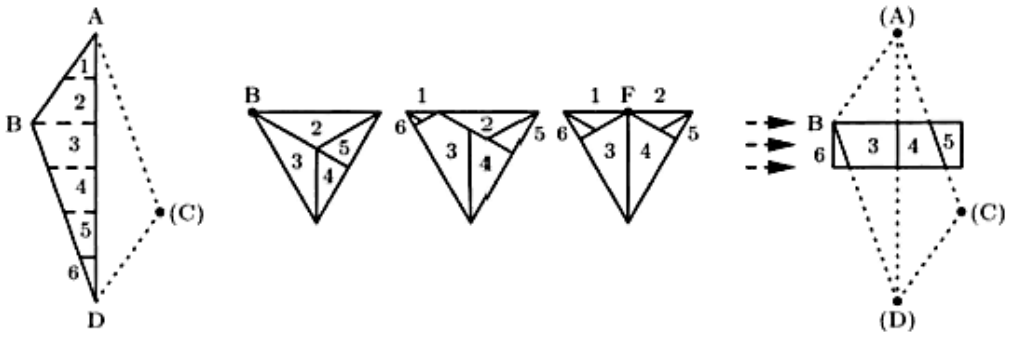


20.5: Hill tetrahedron (first type) to triangular prism

an elegant 3-piece dissection of any Hill tetrahedron of the first type to a triangular prism, where the triangle is equilateral.

Figure 20.5 orients the tetrahedron so that vertex A is directly above vertex D. In this orientation, the projected outline of the tetrahedron, when viewed from above, is an equilateral triangle. Cross sections of Hill's tetrahedron appear in the top row of this figure, with the projected outline of the tetrahedron shown with dotted lines. The bottom row illustrates the dissection itself, with front views of the tetrahedron and prism given on the extreme left and right, respectively. The dissection is exceptionally simple, with just two horizontal slices made, one through vertex B and the other through vertex C. The pieces may be hinged, with pieces 1 and 2 hinged at the level of B, and pieces 2 and 3 hinged at the level of C. The arrows indicate the levels at which cross sections are viewed. The middle of the bottom row gives three cross sections of the prism, at the levels indicated. Hanegraaf also noted that for one of the tetrahedra of Hill's first type, the resulting thickness of the triangular prism exactly equals the side length of a cube of the same volume. In this case, we can project the triangle-to-square dissection through the height of the prism. This gives a 9-piece fully hinged dissection of a particular tetrahedron of Hill's first type to a cube.

Schöbi, a mathematics student at ETH-Zurich, who received his doctorate in 1990, also extended the idea behind the dissection of Figure 20.5 to the Hill tetrahedra of the second and third type: He (1985) gave 4-piece and 6-piece dissections, respectively, for these figures to triangular prisms. Schöbi's dissection for the third



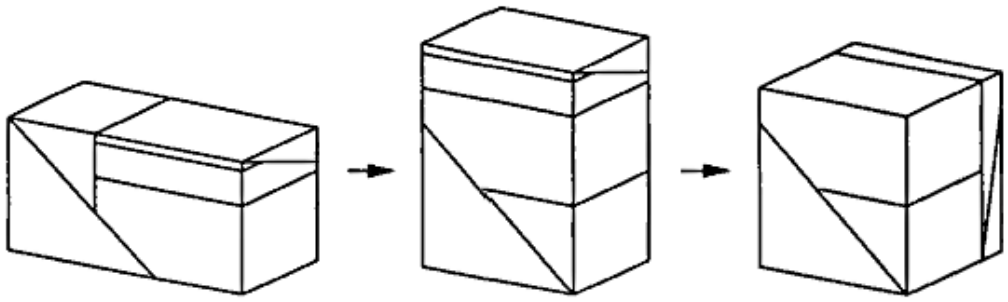
20.6: Hill tetrahedron (third type) to triangular prism

type is in Figure 20.6. An example of Hill's tetrahedra of the third type is on the left, with vertex A directly above vertex D, and with the outline of the tetrahedron of the first type from which it is derived shown with dotted lines. Schöbi sliced the tetrahedron by five horizontal slices into six pieces. Vertex F is at the rear of the right-hand boundary between pieces 3 and 4. The resulting prism is on the right, with three cross sections shown in the middle. The resulting prism is one-half the height of the prism in Figure 20.5.

Puzzle 20.1: Find a 4-piece dissection of Hill's tetrahedron of the second type to a triangular prism.

The preceding examples of tetrahedral dissections arose in response to concerns about demonstrating equivalence of volume with dissection methods. An assumption that underlies the viewpoint of Hill, Hilbert, and Dehn is that dissecting a rectangular solid to a cube is easy. Surprisingly, there seems to have been no explicit mention of such a dissection until William Fitch Cheney (1933) gave an 8-piece dissection of a $(2 \times 1 \times 1)$-rectangular block to a cube. The resulting cube has each dimension equal to $r = \sqrt[3]{2}$. Cheney first converted the $(2 \times 1 \times 1)$-block to an $(r \times r^2 \times 1)$-block, using a P-strip on a face that is a (2×1)-rectangle, to convert it to an $(r \times r^2)$-rectangle. He projected the cuts on this face into the third dimension along planes perpendicular to that face, creating three pieces. Cheney then converted the $(r \times r^2 \times 1)$-block to the cube by a similar operation on a face that is a $(r^2 \times 1)$-rectangle. The cuts from the second operation intersect those of the first, giving a total of eight pieces.

Before Cheney's solution could appear in print, Albert Wheeler (1935) discovered a 7-piece solution. The improved dissection had the same strategy of converting the original block to an $(r \times r^2 \times 1)$-block and then to a cube. However, Wheeler used a P-slide in place of each P-strip. By orienting the second P-slide appropriately,



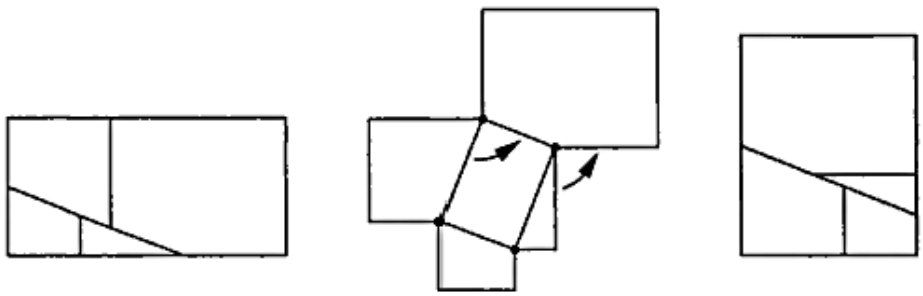
20.7: Hanegraaf's (2 x 1 x 1)-rectangular block to cube

he saved a piece. Care in avoiding intersecting cuts is a key insight, which, it turns out, Wheeler did not exploit fully, for Wheeler's dissection is not minimal, either!

A 6-piece dissection by Anton Hanegraaf (1989) is in Figure 20.7. Hanegraaf began as Wheeler had, by using a P-slide to form the $(r \times r^2 \times 1)$-block. In the second step, Hanegraaf avoided crossing previous cuts by slicing the $(r \times r^2 \times 1)$-block into an $(r \times (r^2-r) \times 1)$-block and an $(r \times r \times 1)$-block. He then used a P-slide to convert the $(r \times (r^2-r) \times 1)$-block to a $(r \times (r-1) \times r)$-block. He rotated the resulting block and placed it next to the $(r \times r \times 1)$-block to give the cube. Although Hanegraaf saved a piece, his dissection is not translational, as was Wheeler's. Hanegraaf's approach works on rectangular blocks spanning a range of different dimensions.

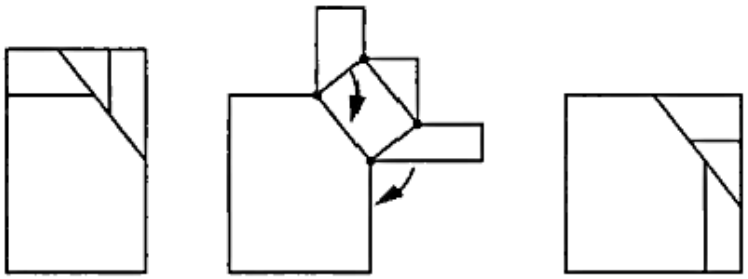
If we do not mind using more pieces, then it is possible to create a dissection that is cyclicly hinged. This was first realized by Michael Goldberg (1966), who used two applications of Harry Lindgren's Q-slide technique. Goldberg claimed that his dissection had ten pieces, but, as Hanegraaf pointed out, it actually had twelve pieces, two of which were very thin triangular prisms. But why quibble, because Hanegraaf (1989) gave a cyclicly hinged dissection that has just seven pieces!

Hanegraaf used the same strategy as Cheney and Wheeler, converting the original block to an $(r \times r^2 \times 1)$-block and then to a cube, but employed his T-slide technique (Figure 11.31). The T-slide is particularly nice when the trapezoid is



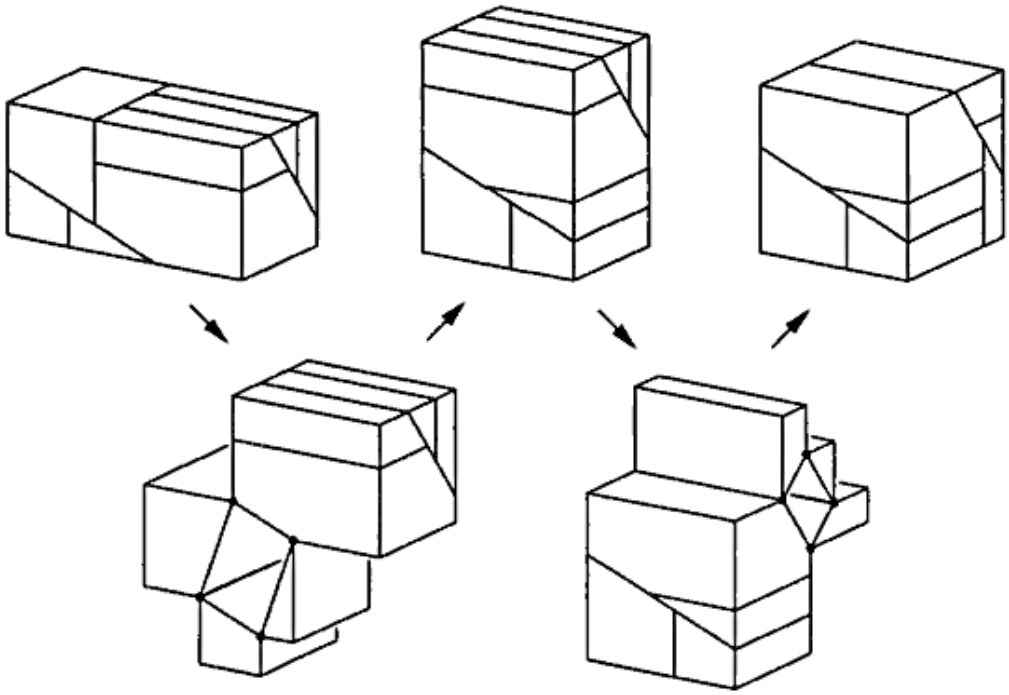
20.8: Hanegraaf's hinged slide

a parallelogram, since there is no cut that crosses the parallelogram along the long dimension. A T-slide takes a (2×1)-rectangle to an $(r \times r^2)$-rectangle in Figure 20.8. A second illustration in Figure 20.9 converts a $(1 \times r^2)$-rectangle to a square.



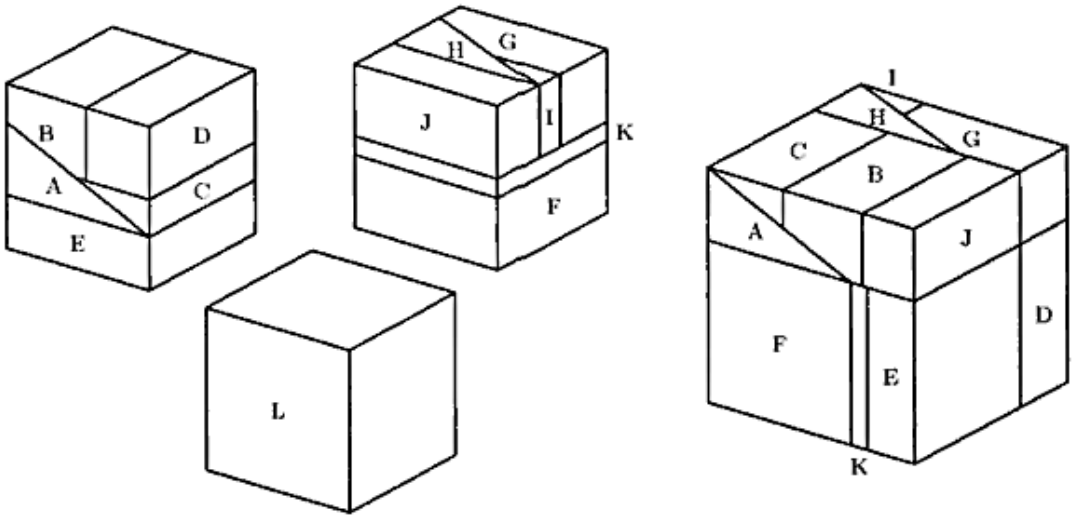
20.9: A second hinged slide by Hanegraaf

Hanegraaf applied the dissections in Figures 20.8 and 20.9 to the appropriate faces of blocks in the solid dissection, with the cuts projecting through the third dimension, as in Cheney's and Wheeler's dissections. The resulting 7-piece dissection, a variant of Hanegraaf's, is illustrated in Figure 20.10. The figure shows the $(2 \times 1 \times 1)$-block on the top left and the cube on the top right, with intermediate steps in the pair of hinged transformations in between.



20.10: Hanegraaf's hinged (2 x 1 x 1)-rectangular block to cube

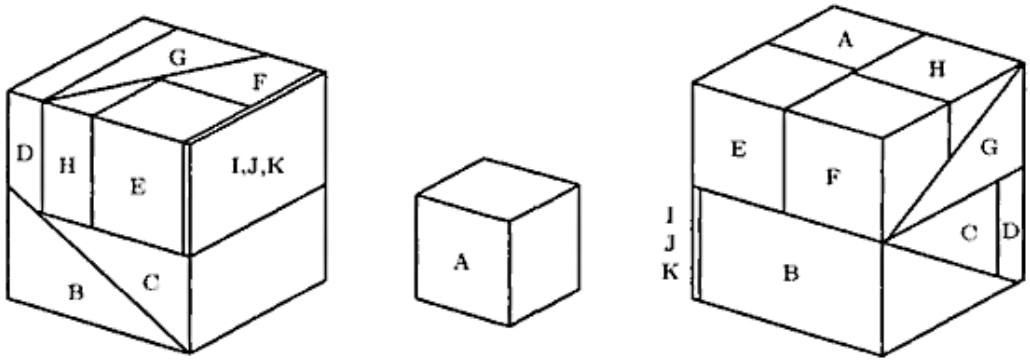
Suppose that we replace the $(2 \times 1 \times 1)$-block by two equal cubes. How many more pieces does it take to dissect the pairs of cubes, rather than the block, to a larger cube? If we chop the $(2 \times 1 \times 1)$-block of Figure 20.7 in half, this gives us an 11-piece dissection. Hanegraaf (1989) found a 10-piece solution, then hid it within Figure 20.15. Can the reader find it?



20.11: Hanegraaf's three cubes to one

Hanegraaf next turned to three cubes to one. His clever 12-piece dissection is shown in Figure 20.11. Let $s = \sqrt[3]{3}$ be the size of each dimension of the resulting cube. The s-cube is assembled from an $(s \times (s-1) \times (s-1))$-block (pieces G, H, and I) and an $(s \times 1 \times (s-1))$-block (pieces A, B, C, and J) on the top, and an $((s-1) \times (s-1) \times s)$-block (piece D), an $((s-1) \times 1 \times 1)$-block (pieces E and K), an $((s-1) \times 1 \times 1)$-block (piece F), and a $(1 \times 1 \times 1)$-block (piece L) on the bottom. Piece L is obscured in the s-cube by the other pieces in Figure 20.11. Hanegraaf used P-slides in two ways: His rearrangement of pieces G, H, and I is standard, but he cut pieces A, B, and C so that they fit around piece D in the 1-cube and also fill in around piece J in the s-cube.

Let's shift from equal to unequal cubes, asking for a physical realization of solutions to $x^3 + y^3 = z^3$, with $x < y$. Anton Hanegraaf's detailed study of these problems identified a number of cases in terms of the relative sizes of x and y. It used two general approaches. The first left the x-cube intact and converted the y-cube to three blocks: an $(x \times x \times (z-x))$-block, an $(x \times (z-x) \times z)$-block, and a $((z-x) \times z \times z)$-block. If $x < y(\sqrt{31/3} - 1)/4 \approx 0.5536y$, then eleven pieces result. We see this case in Figure 20.12, where the x-cube is piece A; the first block above

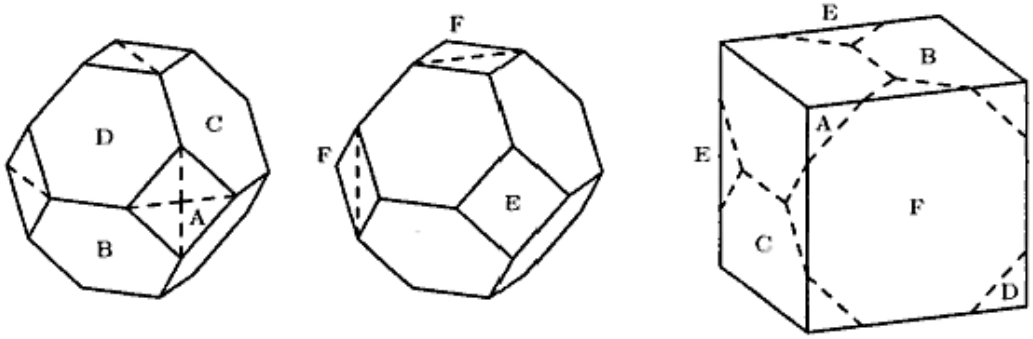


20.12: Cubes for $x^3 + y^3 = z^3$, with $x/y = 1/2$

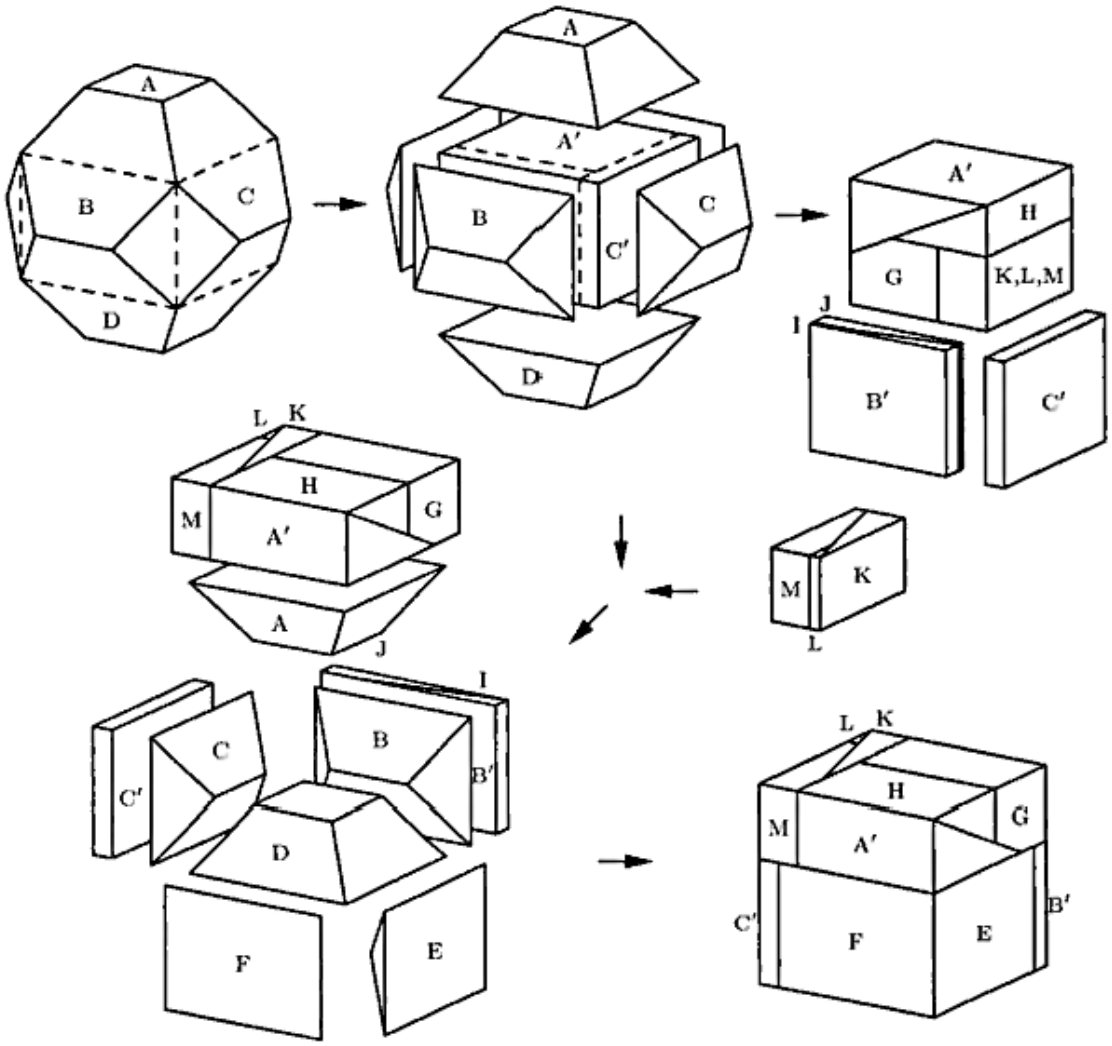
is piece E; the second consists of pieces F, G, and H; and the third the remaining pieces B, C, D, I, J, and K. The last three pieces effect a P-slide from a $((z-x) \times (z(z-y)/y) \times y)$-block in the y-cube to a $((z-x) \times (z-y) \times z)$-block in the z-cube. Our example demonstrates the case when $x/y = 1/2$. Unfortunately, as this ratio gets smaller, pieces I, J, and K get extremely narrow. If $y(\sqrt{31/3} - 1)/4 < x < y(\sqrt{5} - 1)/2 \approx 0.618y$, then twelve pieces result.

The second approach converted the x-cube to a $(z \times z \times (x^3/z^2))$-block, converted the y-cube to a $(z \times z \times (y^3/z^2))$-block, and then set these blocks side by side. This gave a 12-piece dissection for $x > y\sqrt[3]{1/(2\sqrt{2} - 1)}$. For $y(\sqrt{5} - 1)/2 < x < y\sqrt[3]{1/(2\sqrt{2} - 1)}$, either approach used thirteen pieces. In all cases, Hanegraaf used the P-slide in a fashion similar to that in three equal cubes to one. He also surveyed those cases for which a step can replace one or more of the P-slides.

Besides cubes, truncated octahedra are not very hard to dissect, in part because they honeycomb three dimensions. A direct consequence of this honeycomb is a 6-piece dissection of two truncated octahedra to one cube (Figure 20.13), which both David Paterson (1988) and Anton Hanegraaf noted. They cut the first truncated octahedron into four equal pieces by a horizontal slice and a vertical slice. Then they



20.13: Two truncated octahedra to cube



20.14: Hanegraaf's truncated octahedron to cube

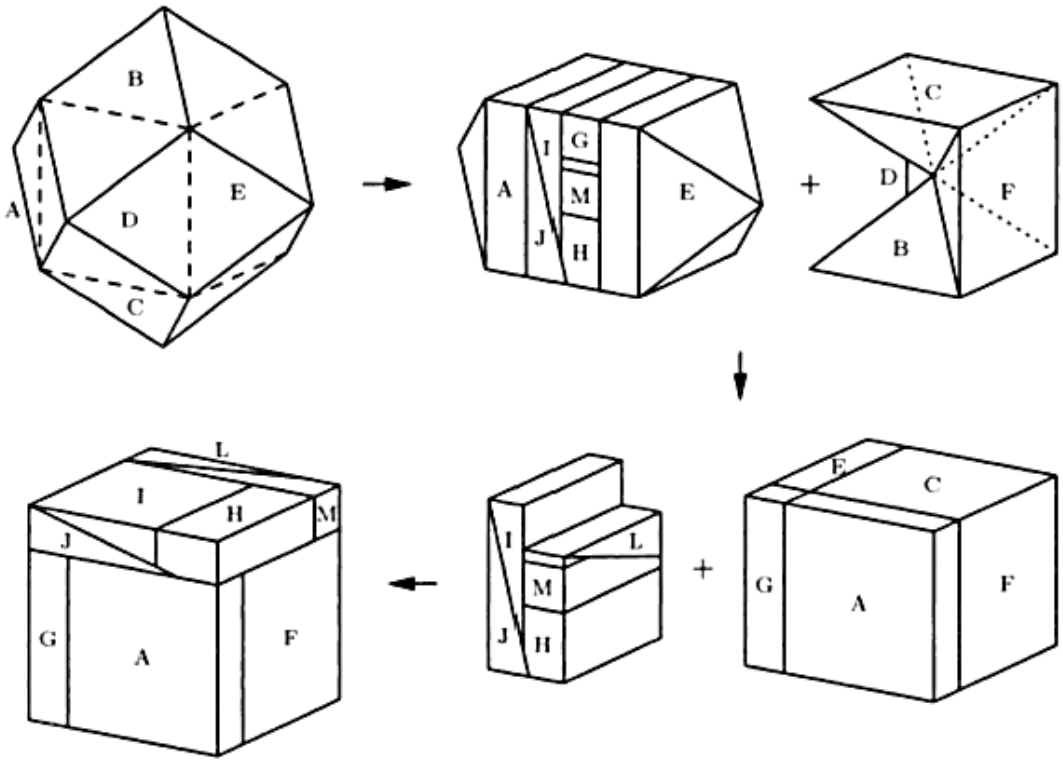
cut the second truncated octahedron into two equal pieces by a slice perpendicular to the planes of the other two slices. The resulting dissection is translational, with reflection and 4-fold rotational symmetry.

More challenging is dissecting one truncated octahedron to one cube. Paterson (1988) gave a 17-piece dissection, in which he used a generalization of the P-strip technique in three dimensions. However, Hanegraaf found a 13-piece dissection, based once again on the P-slide technique.

Hanegraaf's dissection is illustrated in Figure 20.14. First, he cut away the top and bottom (pieces A and D), and also the four sides (pieces C, D, E, and F), as indicated by the dashed edges on the upper left. These cuts leave a rectangular block, and the six pieces form an identical rectangular block, so that the remaining work combines these two blocks into a cube. Then he divided the solid rectangular

block into three blocks by the dashed lines in the second part of the figure. If an edge of the truncated octahedron is of length 1, then piece C′ is $(2\sqrt[6]{2} - 2)$ wide. A P-slide makes the second block the same width and changes its length to $2\sqrt[6]{2}$. The P-slide creates pieces B′, I, and J. Hanegraaf cut the remaining block into six pieces (A′, G, H, K, L, M) via two P-slides, which together form a $(2\sqrt[6]{2} - \sqrt{2})$-wide slice of the resulting cube. We see an exploded view of the cube and the assembled cube at the bottom of the figure. At first it appears that Hanegraaf used sixteen pieces, but the reader can verify that A and A′ need not be separated, and similarly for B and B′, and for C and C′. So Hanegraaf's dissection uses only thirteen pieces.

The rhombic dodecahedron is another three-dimensional figure that is not hard to dissect. In Chapter 2, we discussed how to dissect a rhombic dodecahedron into seven pieces that fit together to form two cubes. Of course, we can then dissect it to one cube, and Anton Hanegraaf found a nifty 13-piece dissection for this case. Hanegraaf's dissection is similar to his dissection of a truncated octahedron to a cube and is illustrated in Figure 20.15. He cut four of the six caps (pieces B, C, D, and F) off the rhombic dodecahedron and arranged them to form two-thirds of a small cube. The remaining two caps, extended with rectangular blocks, form pieces A and E, which fill out the small cube and extend it to fill out a portion of the



20.15: Hanegraaf's rhombic dodecahedron to a cube

desired cube. He cut the remainder of the rhombic dodecahedron into seven pieces (G through M), using two P-slides, to fill out the remaining volume of the desired large cube. The small unlabeled wedge in several of the objects is piece K.

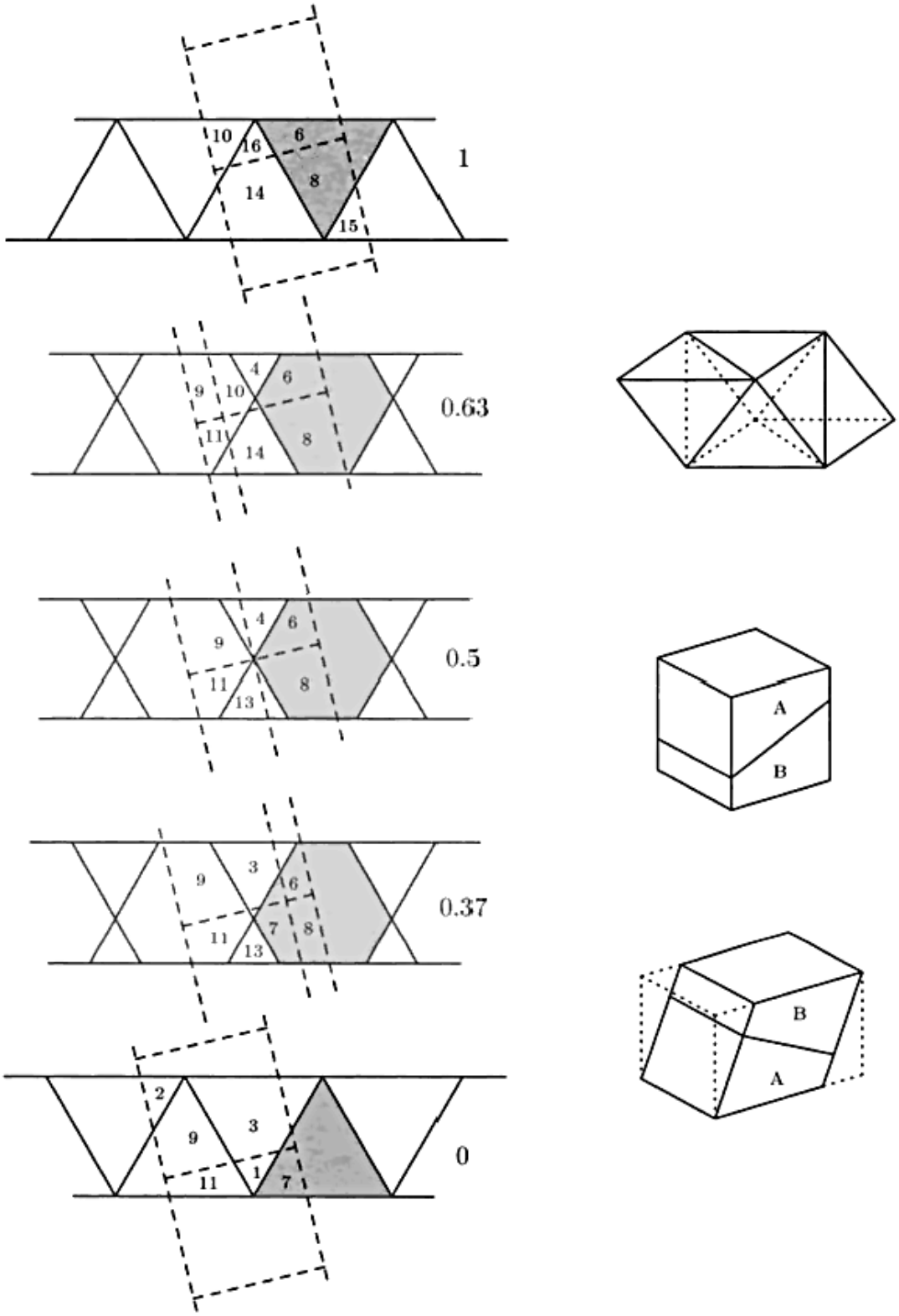
Two tetrahedra plus an octahedron form an element that can honeycomb 3-space. Thus there is a dissection of two tetrahedra and an octahedron to a cube, even though neither alone has a dissection to a cube. Duilio Carpitella (1996) gave a symmetrical 16-piece dissection. The honeycomb element is the parallelopiped shown on the top right in Figure 20.16. In a fashion analogous to forming a strip, we can line up copies of this element, one after another, to form a *post*, whose cross section is a parallelogram.

We can then place posts flush against each other to form a *wall* of uniform thickness. We can shift the posts relative to each other, in the same way that we shift strips when we tessellate the plane. We can also create posts and walls when using a cube as a honeycomb element, but crossposing the two posts is difficult to visualize in three dimensions. Carpitella simplified the approach by slicing the cube so that the two resulting pieces form a parallelopiped whose height equals the thickness of the tetrahedron-octahedron wall. The sliced cube and the resulting parallelopiped appear in the middle right and lower right of Figure 20.16. The dotted lines give the outline of a rectangular box in which the parallelopiped fits. We can now use this parallelopiped as a honeycomb element to form a post, and then a wall.

To produce the dissection, crosspose the tetrahedron-octahedron post with the parallelopiped post, while constraining the walls induced by the two posts to

Anton Hanegraaf was born in 1931 in Diessen, the Netherlands. He completed studies in civil engineering at the technological college in 's-Hertogenbosch in 1952. After serving two years of military service in the Engineering Corps, he worked for three years as a structural designer for an architectural firm, and for several more as a free-lance structural engineering consultant. From 1962 until his retirement in 1992, he was an employee of ABT, a firm of structural engineering consultants. His work involved both practical and theoretical research, the development of structural analysis methods, the study of the behavior of structural materials, the study of structural shapes, and the study of morphological aspects of space enclosure.

Anton's recreational pursuits include mechanical and mathematical puzzles, mathematical art, and listening to music and reading a wide variety of books. His mathematical interests include mathematical models, mechanisms and linkages, tilings, polyhedra, and sliding piece and rotational puzzles. He was editor of *Cubism for Fun* from 1987 to 1993. He plans to publish five more booklets in his series *Polyhedral Dissections*.



20.16: Carpitella's one octahedron plus two tetrahedra to one cube

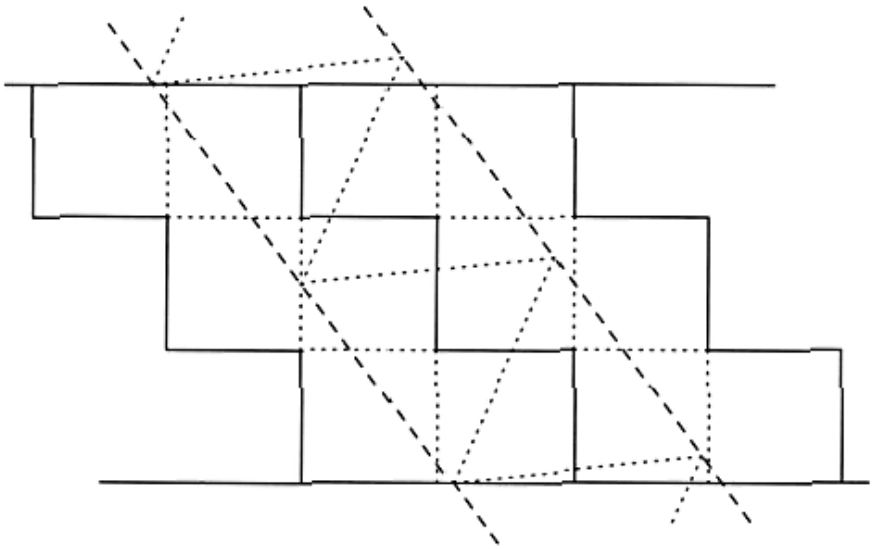
coincide. Since this is still challenging to visualize, we rely on the cross sections shown on the left in Figure 20.16. Each of the five cross sections gives the crossposition at a different height from the bottom of the wall: 1, 0.63, 0.5, 0.37, 0, where 1 is at the top, 0.5 is at the middle, and 0 is at the bottom. One of the octahedra in each of the cross sections is shaded.

We can number the sixteen pieces in the dissection, with the first four pieces forming one tetrahedron, the last four forming the second tetrahedron, and each piece and its mirror image having numbers that sum to 17. Some of the pieces are either too small or too thin to be labeled in each cross section. At height 1, the unlabeled triangular piece adjacent to pieces 10 and 14 is piece 12. At height 0.63, the unlabeled pieces are 16, 12, and 13, in clockwise order around piece 14. At height 0.5, the unlabeled pieces adjacent to pieces 4 and 13 are, respectively, 3 and 14. We can infer unshown labels at levels 0.37 and 0 in a fashion similar to those on levels 0.63 and 1.

Carpitella (1996) investigated a variety of three-dimensional dissections that rely on three-dimensional analogues of plain-strip dissections. When necessary, a three-dimensional element for one strip may be sliced through a plane to give an appropriate thickness in the resulting element. Carpitella employed this approach to give 14-piece symmetrical dissections of a rhombic dodecahedron to a cube, and of a truncated octahedron to a cube. He also found a 28-piece dissection of a rhombic dodecahedron to a truncated octahedron.

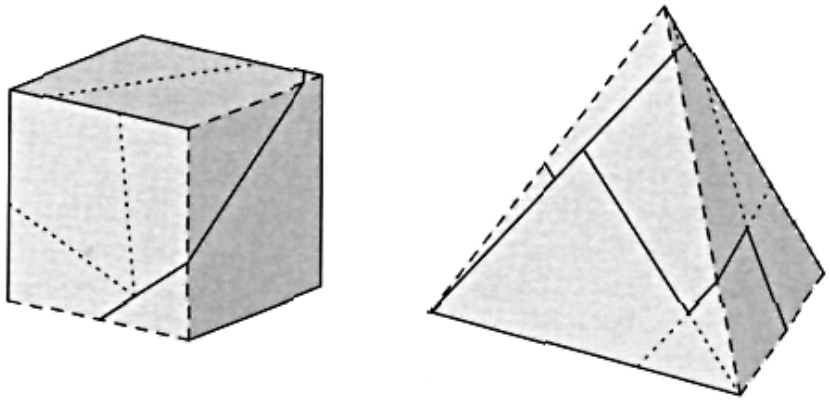
Our excursion into three-dimensional dissections has been subject to rather severe limitations on which dissections are possible. However, Gavin Theobald posed a natural (and in hindsight obvious) antidote: Dissect the surfaces instead! This leads to many surprising results, one of which we will see here. Theobald discovered a number of 2-piece dissections, based on the flexibility of cutting and opening out the surfaces of three-dimensional figures. (He also discovered some other surface dissections that used more than two pieces.)

Duilio Carpitella was born in Trapani, Sicily, Italy, in 1957. After attending the First Artistic High School in Rome, he enrolled at the Faculty of Architecture at the University of Rome ("La Sapienza"). He completed his studies in 1983 and worked in an architectural office for four years, before opening a studio in Rome with two colleagues to work with illustration. A major interest of Carpitella's has been painting; his work derives from an analysis of M. C. Escher's work on tessellations. He has participated in exhibitions of work inspired by Escher and on mathematical imagery. He also enjoys inventing games.



20.17: Theobald's crossposition of cube and tetrahedron surfaces

Theobald cut the surface of a tetrahedron along two opposite edges to give a band, then with one more cut gave a parallelogram. In addition, Theobald unfolded the surface of a cube in a zigzag fashion to give a P-strip element. The associated crossposition is in Figure 20.17, where the solid lines show the cube's P-strip element, the dashed edges show the boundary of the tetrahedron P-strip, and the dotted lines show the folds. A view of the 2-piece dissection is in Figure 20.18. In the cube, solid lines indicate the cuts, dashed lines the edges that are not cut, and dotted lines the folds from the tetrahedron. We can use the same convention in the tetrahedron, except that there the dotted lines represent folds from the cube.



20.18: Theobald's surface of cube to surface of tetrahedron